Connection between type B (or C) and F factorizations and construction of algebras

2002 J. Phys. A: Math. Gen. 35 2857

(http://iopscience.iop.org/0305-4470/35/12/310)

Institute of Physics Publishing Journal of Physics A: Mathematical and General
J. Phys. A: Math. Gen. **35** (2002) 2857–2865 PII: S0305-4470(02)29044-7

Connection between type B (or C) and F factorizations and construction of algebras

A Del Sol Mesa[1] **and C Quesne**[2,3]

[1] Bioinformatics Unit, Department of Computer Science, University College London, Gower Street, London WC1E 6BT, UK
[2] Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles, Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium

E-mail: A.DelSolMesa@cs.ucl.ac.uk and cquesne@ulb.ac.be

Received 25 September 2001, in final form 4 February 2002
Published 15 March 2002
Online at stacks.iop.org/JPhysA/35/2857

Abstract
In a recent paper (Del Sol Mesa A and Quesne C 2000 *J. Phys. A: Math. Gen.* **33** 4059), we started a systematic study of the connections among different factorization types, suggested by Infeld and Hull, and their consequences for the construction of algebras. We devised a general procedure for constructing satellite algebras for all the Hamiltonians admitting a type E factorization by using the relationship between type A and E factorizations. Here we complete our analysis by showing that for Hamiltonians admitting a type F factorization, a similar method, starting from either type B or type C factorization, leads to other types of algebras. We therefore conclude that the existence of satellite algebras is a characteristic property of type E factorizable Hamiltonians. Our results are illustrated with the detailed discussion of the Coulomb problem.

PACS numbers: 02.10.De, 02.20.Sv, 02.30.−f

1. Introduction

In a recent paper [1] (henceforth referred to as I and whose equations will be quoted by their number preceded by I), we investigated the role of the factorization method [2, 3] in the construction of a new class of symmetry algebras, called *satellite algebras*, introduced in a previous paper [4]. Such algebras, generalizing the so-called potential algebras [5, 6], depend upon some auxiliary variables and connect among themselves wavefunctions belonging to different satellite potentials and corresponding to different energy eigenvalues (instead of the same ones). We also devised a general procedure for determining an $so(2,2) \simeq su(1,1) \oplus su(1,1)$ or $so(2,1) \simeq su(1,1)$ satellite algebra for all the Hamiltonians

[3] Directeur de recherches FNRS.

admitting a type E factorization by using the known relationship between type A and E factorizations and an algebraization similar to that used in the construction of potential algebras.

The purpose of the present paper is to apply a similar procedure to the Hamiltonians admitting a type F factorization by starting from type B or C factorization. Let us recall that in such cases, the functions $r(x,m)$, $k(x,m)$ and $L(m)$ entering the factorization method formalism (see section 2 of I for a summary of the latter) are given in terms of some constants a, b, c, d and q by

$$\text{Type B:}\qquad r(x,m) = -\mathrm{d}^2\,\mathrm{e}^{2ax} + 2ad\left(m+c+\tfrac{1}{2}\right)\mathrm{e}^{ax} \tag{1.1}$$

$$k(x,m) = \mathrm{d}\,\mathrm{e}^{ax} - (m+c)a \tag{1.2}$$

$$L(m) = -a^2(m+c)^2 \tag{1.3}$$

$$\text{Type C:}\qquad r(x,m) = -\frac{(m+c)(m+c+1)}{x^2} - \frac{1}{4}b^2x^2 + b(m-c) \tag{1.4}$$

$$k(x,m) = \frac{m+c}{x} + \frac{1}{2}bx \tag{1.5}$$

$$L(m) = -2bm + \tfrac{1}{2}b \tag{1.6}$$

$$\text{Type F:}\qquad r(x,m) = -\frac{2q}{x} - \frac{m(m+1)}{x^2} \tag{1.7}$$

$$k(x,m) = \frac{m}{x} + \frac{q}{m} \tag{1.8}$$

$$L(m) = -\frac{q^2}{m^2} \tag{1.9}$$

respectively.

Since the precise relationship between type F and type B or C factorizations is not detailed in [2], we first derive it and then construct the corresponding algebra in sections 2 and 3. The general results are illustrated by the example of the Coulomb problem in section 4. Section 5 contains the conclusion.

2. Case of type B and F factorizations

Let us consider the most general second-order differential equation admitting a type F factorization. From equations (I2.1) and (1.7), it is given by

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2q}{r} - \frac{m(m+1)}{r^2} + \lambda\right)\psi(r) = 0 \tag{2.1}$$

where the variable x is changed to r and $\psi = Y_l^m$ denote the normalized eigenfunctions corresponding to the discrete eigenvalues $\lambda = \lambda_l$. Since we want to apply the theory to Hermitian Hamiltonians with bound states, we restrict ourselves to the case where q is some negative real constant. Then $L(m)$, given in (1.9), is an increasing function of m so that the problem is of class I, $m = 0, 1, \ldots, l$, and

$$\lambda = L(l+1) = -\frac{q^2}{(l+1)^2}. \tag{2.2}$$

In the corresponding $k(x,m)$, given in (1.8), m occurs in the denominator, hence a straightforward algebraization of the ladder operators is impossible.

To carry out such an algebraization, one may transform the type F factorizable equation (2.1) into a type B or C factorization equation. In the present section, we consider the former alternative, given by

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \bar{d}^{\,2}\,\mathrm{e}^{2\bar{a}x} + 2\bar{a}\bar{d}\left(\bar{m} + \bar{c} + \frac{1}{2}\right)\mathrm{e}^{\bar{a}x} + \bar{\lambda}\right]\bar{\psi}(x) = 0 \tag{2.3}$$

where a bar is put on top of all the constants to distinguish them from those used for type F factorization and the normalized eigenfunctions $\bar{Y}_{\bar{l}}^{\bar{m}}$, corresponding to the eigenvalues $\bar{\lambda} = \bar{\lambda}_{\bar{l}}$, are denoted by $\bar{\psi}$. From equations (I2.4), (1.2) and (1.3), it follows that the associated ladder operators $\bar{H}^{\pm}(\bar{m})$, which depend linearly on $\bar{m}$, and the real constant $\bar{L}(\bar{m})$ can be written as

$$\bar{H}^{\pm}(\bar{m}) = \pm\frac{\mathrm{d}}{\mathrm{d}x} + \bar{d}\,\mathrm{e}^{\bar{a}x} - (\bar{m} + \bar{c})\bar{a} \tag{2.4}$$

$$\bar{L}(\bar{m}) = -\bar{a}^2(\bar{m} + \bar{c})^2. \tag{2.5}$$

By performing a change of variable and function

$$r = \mathrm{e}^{\bar{a}x} \qquad \psi(r) = \mathrm{e}^{\bar{a}x/2}\bar{\psi}(x) \tag{2.6}$$

equation (2.1) is transformed into an equation of type (2.3),

$$\left[\frac{\mathrm{d}^{\,2}}{\mathrm{d}x^2} + \lambda\bar{a}^2\mathrm{e}^{2\bar{a}x} - 2q\bar{a}^2\mathrm{e}^{\bar{a}x} - \left(m + \frac{1}{2}\right)^2\bar{a}^2\right]\bar{\psi}(x) = 0. \tag{2.7}$$

Comparison between (2.3) and (2.7) shows that the parameters and eigenvalues of both factorization types are connected through the relations

$$\frac{\bar{d}^{\,2}}{\bar{a}^2} = -\lambda \tag{2.8}$$

$$\frac{\bar{d}}{\bar{a}}\left(\bar{m} + \bar{c} + \frac{1}{2}\right) = -q \tag{2.9}$$

$$\frac{\bar{\lambda}}{\bar{a}^2} = -\left(m + \frac{1}{2}\right)^2. \tag{2.10}$$

From (2.6), it is clear that only the real values of $\bar{a}$ are to be considered here. Hence from (2.5), it results that the type B problem is of class II, $\bar{m} = \bar{l}, \bar{l} + 1, \ldots,$ and

$$\bar{\lambda} = \bar{L}(\bar{l}) = -\bar{a}^2(\bar{l} + \bar{c})^2. \tag{2.11}$$

Equation (2.9) and the assumption $-q \in \mathbb{R}^+$ imply that $\bar{m} + \bar{c} + \frac{1}{2}$ and $\bar{d}/\bar{a}$ both belong to $\mathbb{R}^+$ or $\mathbb{R}^-$. The latter alternative is excluded because $\bar{m}$ may take arbitrary large positive values. Moreover, we may always assume that $\bar{a} \in \mathbb{R}^+$ since changing the sign of $\bar{a}$ can be compensated by the change of variable $x \to -x$. We therefore conclude that the parameters of type B and F factorizations may be restricted to values such that $\bar{a}, \bar{d}, \bar{m} + \bar{c} + \frac{1}{2}$ and $-q \in \mathbb{R}^+$.

In such a case, equations (2.8)–(2.10), together with (2.2) and (2.11), lead to the relations

$$\frac{\bar{d}}{\bar{a}} = -\frac{q}{l+1} = \sqrt{-\lambda} \qquad \bar{m} + \bar{c} = l + \frac{1}{2} \qquad \bar{l} + \bar{c} = m + \frac{1}{2}. \tag{2.12}$$

On using (2.6) and (2.12), the type B ladder operators (2.4) lead to ladder operators for the original type F eigenfunctions ψ,

$$\begin{aligned}\tilde{H}^{\pm}(l) &\equiv \mathrm{e}^{\bar{a}x/2}\bar{H}^{\pm}(\bar{m})\mathrm{e}^{-\bar{a}x/2} \\ &= \bar{a}\left[\pm r\frac{\mathrm{d}}{\mathrm{d}r} + \sqrt{-\lambda}r - \left(l + \frac{1}{2} \pm \frac{1}{2}\right)\right].\end{aligned} \tag{2.13}$$

From these ladder operators, we can get Lie algebra generators by introducing an auxiliary variable $\eta \in [0, 2\pi)$ and extended eigenfunctions defined by

$$\Psi_t(r, \eta) = (2\pi)^{-1/2}\mathrm{e}^{\mathrm{i}t\eta}\psi(r) \tag{2.14}$$

where

$$t \equiv l + 1. \tag{2.15}$$

Since

$$T_0 = -\mathrm{i}\frac{\partial}{\partial \eta} \tag{2.16}$$

is such that

$$T_0 \Psi_t(r, \eta) = t \Psi_t(r, \eta) \tag{2.17}$$

we may replace t by $-\mathrm{i}\partial/\partial\eta$ when dealing with extended eigenfunctions. By combining the transformation

$$\bar{a}^{-1} \mathrm{e}^{\pm \mathrm{i}\eta} \tilde{H}^{\mp} \left(l + \tfrac{1}{2} \pm \tfrac{1}{2}\right) \to T_{\pm} \tag{2.18}$$

with this substitution, we obtain

$$T_{\pm} = \mathrm{e}^{\pm \mathrm{i}\eta} \left(\mp r \frac{\partial}{\partial r} + \mathrm{i} \frac{\partial}{\partial \eta} + \sqrt{-\lambda} r \right). \tag{2.19}$$

It is straightforward to check that the operators T_0, T_+, T_- close an $su(1, 1) \simeq so(2, 1)$ Lie algebra, i.e.

$$[T_0, T_{\pm}] = \pm T_{\pm} \qquad [T_+, T_-] = -2T_0 \tag{2.20}$$

and that the Casimir operator of the latter can be written as

$$\begin{aligned} C &\equiv -T_+ T_- + T_0(T_0 - 1) \\ &= r^2 \left(\frac{\partial^2}{\partial r^2} - 2\mathrm{i} \frac{\sqrt{-\lambda}}{r} \frac{\partial}{\partial \eta} + \lambda \right). \end{aligned} \tag{2.21}$$

From (2.1), (2.2), (2.15), (2.16) and (2.17), it results that the action of C on the extended eigenfunctions (2.14) is given by

$$C \Psi_t(r, \eta) = m(m + 1) \Psi_t(r, \eta). \tag{2.22}$$

The (nonunitary) $su(1, 1)$ irreducible representations spanned by $\Psi_t(r, \eta)$ may therefore be characterized by m and such functions may be denoted by $\Psi_t^{(m)}(r, \eta)$. It should be stressed that the $su(1, 1)$ generators $T_{\pm}$ change t to $t \pm 1$, while leaving m and λ unchanged. From the first relation in (2.12), it is clear that q becomes $q' = q(l + 1 \pm 1)/(l + 1) = q(t \pm 1)/t$. Hence $T_{\pm}$ connect among themselves eigenfunctions corresponding to different potentials but the same energy eigenvalue. We therefore conclude that the $su(1, 1)$ algebra resulting from type B factorization is a potential algebra.

3. Case of type C and F factorizations

Let us now turn to the latter alternative, i.e. transforming the type F factorizable equation (2.1) into a type C factorizable equation. For such a purpose, we shall take advantage of the relation between types F and B already established in section 2 and of the equivalence between types B and C noted in [2].

More specifically, the type B factorizable equation (2.3) can be transformed into a type C factorizable equation,

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{(\hat{m} + \hat{c})(\hat{m} + \hat{c} + 1)}{y^2} - \frac{1}{4}\hat{b}^2 y^2 + \hat{b}(\hat{m} - \hat{c}) + \hat{\lambda} \right] \hat{\psi}(y) = 0 \tag{3.1}$$

where the normalized eigenfunctions $\hat{Y}_{\hat{l}}^{\hat{m}}$ corresponding to the eigenvalues $\hat{\lambda} = \hat{\lambda}_{\hat{l}}$ are denoted by $\hat{\psi}$, by the change of variable and function

$$x = \frac{2}{\bar{a}} \ln \frac{y}{2} \qquad \bar{\psi}(x) = y^{-1/2} \hat{\psi}(y). \tag{3.2}$$

We indeed obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{(4\bar{\lambda}/\bar{a}^2) + \frac{1}{4}}{y^2} - \frac{\bar{d}^2}{4\bar{a}^2} y^2 + \frac{2\bar{d}}{\bar{a}} \left(\bar{m} + \bar{c} + \frac{1}{2} \right) \right] \hat{\psi}(y) = 0 \tag{3.3}$$

which coincides with (3.1) provided

$$(\hat{m} + \hat{c})(\hat{m} + \hat{c} + 1) = -\frac{4\bar{\lambda}}{\bar{a}^2} - \frac{1}{4} \tag{3.4}$$

$$\hat{b}^2 = \frac{\bar{d}^2}{\bar{a}^2} \tag{3.5}$$

$$\hat{b}(\hat{m} - \hat{c}) + \hat{\lambda} = \frac{2\bar{d}}{\bar{a}} \left(\bar{m} + \bar{c} + \frac{1}{2} \right). \tag{3.6}$$

Note also that the ladder operators $\hat{H}^{\pm}(\hat{m})$ and real constants $\hat{L}(\hat{m})$ corresponding to (3.1) are given by

$$\hat{H}^{\pm}(\hat{m}) = \pm \frac{\mathrm{d}}{\mathrm{d}y} + \frac{\hat{m} + \hat{c}}{y} + \frac{1}{2} \hat{b} y \tag{3.7}$$

$$\hat{L}(\hat{m}) = -2\hat{b}\hat{m} + \tfrac{1}{2}\hat{b}. \tag{3.8}$$

In solving equations (3.4)–(3.6), there are in principle two indeterminate signs, namely those of $\hat{b}$ and $\hat{m} + \hat{c} + \frac{1}{2}$. From (3.8), it results that the former determines whether the type C problem is of class I or II. For definiteness sake, we shall assume here that the former alternative holds true, so that $\hat{b} \in \mathbb{R}^-$, $\hat{m} = 0, 1, \ldots, \hat{l}$, and

$$\hat{\lambda} = \hat{L}(\hat{l} + 1) = -\hat{b} \left(2\hat{l} + \tfrac{3}{2} \right). \tag{3.9}$$

The other case can be treated in a similar way.

Equations (2.11), (3.4)–(3.6) and (3.9) then lead to

$$\hat{m} + \hat{c} + \tfrac{1}{2} = 2\epsilon(\bar{l} + \bar{c}) \qquad \hat{b} = -\frac{\bar{d}}{\bar{a}} \qquad \hat{l} + \hat{c} + \tfrac{1}{2} = \bar{m} + \bar{c} + \epsilon(\bar{l} + \bar{c}) \tag{3.10}$$

where $\epsilon = \pm 1$. Note that both signs of $\hat{m} + \hat{c} + \frac{1}{2}$ are allowed due to the finite range of values of $\hat{m}$ characteristic of class I problems.

It now remains to combine (2.6) with (3.2) to transform the type F factorizable equation (2.1) into a type C factorizable equation. The result reads

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{\left(2m + \frac{1}{2}\right)\left(2m + \frac{3}{2}\right)}{y^2} - \frac{q^2}{4(l+1)^2} y^2 - 2q \right] \hat{\psi}(y) = 0 \tag{3.11}$$

the relations between parameters of both factorization types being

$$\hat{b} = \frac{q}{l+1} = -\sqrt{-\lambda} \qquad \hat{m} + \hat{c} = \epsilon(2m+1) - \tfrac{1}{2} \qquad \hat{l} + \hat{c} = l + \epsilon \left(m + \tfrac{1}{2} \right). \tag{3.12}$$

In such a process, the type C ladder operators (3.7) become ladder operators for the original type F eigenfunctions ψ,

$$\check{H}^{\pm}(l, m) \equiv \mathrm{e}^{\bar{a}x/2} y^{-1/2} \hat{H}^{\pm}(\hat{m}) y^{1/2} \, \mathrm{e}^{-\bar{a}x/2} \tag{3.13}$$

or more explicitly

$$\check{H}_1^{\pm}(l,m) = \sqrt{r}\left(\pm\frac{\mathrm{d}}{\mathrm{d}r} + \frac{2m+\frac{1}{2}\mp\frac{1}{2}}{2r} - \sqrt{-\lambda}\right) \tag{3.14}$$

$$\check{H}_2^{\pm}(l,m) = \sqrt{r}\left(\pm\frac{\mathrm{d}}{\mathrm{d}r} - \frac{2m+\frac{3}{2}\pm\frac{1}{2}}{2r} - \sqrt{-\lambda}\right) \tag{3.15}$$

according to whether we choose $\epsilon = +1$ or $\epsilon = -1$. Since the operators (3.7) leave $\hat{l}$ fixed while changing $\hat{m}$ into $\hat{m} \mp 1$, it follows from (3.12) that the transformed operators (3.14) and (3.15) change both l and m to $l \mp \frac{1}{2}$, $m \pm \frac{1}{2}$ and $l \pm \frac{1}{2}$, $m \pm \frac{1}{2}$, respectively[4].

This time Lie algebras can be obtained by introducing two auxiliary variables α, $\beta \in [0, 2\pi)$ and extended eigenfunctions

$$\Psi_{\mu,\nu}(r,\alpha,\beta) = (2\pi)^{-1}\mathrm{e}^{\mathrm{i}(\mu\alpha+\nu\beta)}\psi(r) \tag{3.16}$$

where

$$\mu \equiv l - m \qquad \nu \equiv l + m + 1. \tag{3.17}$$

By replacing μ and ν by $-\mathrm{i}\partial/\partial\alpha$ and $-\mathrm{i}\partial/\partial\beta$, respectively, and making the transformations

$$(2\sqrt{-\lambda})^{-1/2}\mathrm{e}^{\pm\mathrm{i}\alpha}\check{H}_1^{\pm}\left(l-\tfrac{1}{4}\pm\tfrac{1}{4}, m+\tfrac{1}{4}\mp\tfrac{1}{4}\right) \to A_{\pm} \tag{3.18}$$

$$(2\sqrt{-\lambda})^{-1/2}\mathrm{e}^{\pm\mathrm{i}\beta}\check{H}_2^{\pm}\left(l-\tfrac{1}{4}\pm\tfrac{1}{4}, m-\tfrac{1}{4}\pm\tfrac{1}{4}\right) \to B_{\pm} \tag{3.19}$$

we obtain

$$A_{\pm} = \frac{1}{\sqrt{2\sqrt{-\lambda}}}\,\mathrm{e}^{\pm\mathrm{i}\alpha}\sqrt{r}\left[\pm\frac{\partial}{\partial r} + \frac{1}{2r}\left(\mathrm{i}\frac{\partial}{\partial\alpha} - \mathrm{i}\frac{\partial}{\partial\beta} \mp 1\right) - \sqrt{-\lambda}\right] \tag{3.20}$$

$$B_{\pm} = \frac{1}{\sqrt{2\sqrt{-\lambda}}}\,\mathrm{e}^{\pm\mathrm{i}\beta}\sqrt{r}\left[\pm\frac{\partial}{\partial r} - \frac{1}{2r}\left(\mathrm{i}\frac{\partial}{\partial\alpha} - \mathrm{i}\frac{\partial}{\partial\beta} \pm 1\right) - \sqrt{-\lambda}\right] \tag{3.21}$$

where we note that $A_{\pm}$ and $B_{\pm}$ only differ by the substitutions $\alpha \leftrightarrow \beta$, $\partial/\partial\alpha \leftrightarrow \partial/\partial\beta$.

Contrary to what happens either for type E factorization or for type F factorization when starting from a type B factorization, the operators $A_{\pm}$ and $B_{\pm}$ do not belong to $su(1,1)$ algebras, but instead close two commuting Heisenberg–Weyl algebras:

$$[A_-, A_+] = [B_-, B_+] = I \qquad [A_{\pm}, B_{\pm}] = [A_{\pm}, B_{\mp}] = 0. \tag{3.22}$$

From (3.16)–(3.19), it is obvious that $A_{\pm}$ (respectively $B_{\pm}$) change μ to $\mu \pm 1$ (respectively ν to $\nu \pm 1$), while leaving ν (respectively μ) and λ unchanged. As a consequence, the parameter q is changed to $q' = q\left(l+1\pm\frac{1}{2}\right)\big/(l+1) = q(\mu+\nu+1\pm 1)/(\mu+\nu+1)$.

The interpretation of the $w(1)\oplus w(1)$ algebra obtained in the present section will become clearer when illustrated on the example of the Coulomb problem in the next section.

4. The Coulomb problem

In units wherein $\hbar = \mu = e = 1$, the radial wavefunction for an electron in a Coulomb potential satisfies the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{L(L+1)}{r^2} + \frac{2Z}{r} + 2E\right)R(r) = 0 \tag{4.1}$$

[4] Noninteger values of l and m, forbidden in the original formulation of the factorization method [2], are allowed in its generalization by Cariñena and Ramos [3]. The occurrence of half-integers here is compatible with this extended theory.

where L is the orbital angular momentum and Z the atomic number. The negative-energy eigenvalues and corresponding eigenfunctions are given by [7]

$$E_n = -\frac{Z^2}{2n^2} \qquad n = n_r + L + 1 \tag{4.2}$$

and

$$R_{nL}(r) = N_{nL}\mathrm{e}^{-\frac{1}{2}\rho}\rho^L L_{n-L-1}^{(2L+1)}(\rho) \qquad \rho \equiv \gamma r \qquad \gamma \equiv \frac{2Z}{n} \tag{4.3}$$

where $n_r, L = 0, 1, 2, \ldots, N_{nL}$, is some normalization coefficient, and $L_n^{(\alpha)}(x)$ is a generalized Laguerre polynomial[5].

By setting $R(r) = S(r)/r$, equation (4.1) can be rewritten in a form similar to (2.1) with

$$q = -Z \qquad m = L \qquad \lambda = 2E \qquad \psi(r) = S(r). \tag{4.4}$$

Here $q \in \mathbb{R}^-$ as assumed in sections 2 and 3. Comparing the expressions of λ resulting from (2.2) and from (4.2) and (4.4), we obtain the relation

$$l = n - 1 = n_r + L \tag{4.5}$$

between the eigenvalue labels n and l, resulting from the resolution of the Schrödinger equation and the factorization method, respectively.

Considering first the mapping of (4.1) onto a type B factorizable equation, we get from (2.15) and (4.5)

$$t = n. \tag{4.6}$$

By using (4.3), (4.4), (4.6) and the results of [7] and [8], the corresponding extended eigenfunctions (2.14) can be written as

$$\Psi_t^{(m)}(r, \eta) = (2\pi)^{-1/2} N_t^{(m)} \mathrm{e}^{\mathrm{i}t\eta}\mathrm{e}^{-\frac{1}{2}\rho}\rho^{m+1} L_{t-m-1}^{(2m+1)}(\rho) \qquad \rho = \gamma r \tag{4.7}$$

where

$$N_t^{(m)} = \left(\frac{\gamma(t-m-1)!}{2t(t+m)!}\right)^{1/2}. \tag{4.8}$$

When acting on such extended eigenfunctions, the $su(1, 1)$ generators $T_\pm$ of equation (2.19) become

$$T_\pm = \mathrm{e}^{\pm\mathrm{i}\eta}\left(\mp\rho\frac{\partial}{\partial\rho} + \mathrm{i}\frac{\partial}{\partial\eta} + \frac{1}{2}\rho\right). \tag{4.9}$$

After some calculations using well-known properties of generalized Laguerre polynomials [8], we obtain

$$T_\pm\Psi_t^{(m)} = -\left(\frac{(t\pm 1)(t\mp m)(t\pm m\pm 1)}{t}\right)^{1/2}\Psi_{t\pm 1}^{(m)} \tag{4.10}$$

which, together with (2.17), give the action of the $su(1, 1)$ generators in the Coulomb case. The conserved quantity is the angular momentum L. The operators $T_\pm$ change n to $n \pm 1$ (or n_r to $n_r \pm 1$) and Z to $Z' = Z(n \pm 1)/n$, thus leaving the energy unchanged. Such transitions are relevant to the theory of hydrogen-like ions whenever the ratio Z/n is an integer. As far as we know, this $su(1, 1)$ potential algebra for the Coulomb problem has been noted nowhere else, although it can be easily seen to be equivalent to the supersymmetric analysis of Haymaker and Rau [9].

[5] Contrary to what is done in [7], here we use the conventional definition [8] of generalized Laguerre polynomials.

Considering next the mapping of (4.1) onto a type C factorizable equation, we get from (3.17), (4.4) and (4.5)

$$\mu = n - L - 1 = n_r \qquad \nu = n + L = n_r + 2L + 1. \tag{4.11}$$

The extended eigenfunctions are not given by (4.7) anymore, but instead by

$$\Psi_{\mu,\nu}(r,\alpha,\beta) = (2\pi)^{-1} N_{\mu,\nu}\, \mathrm{e}^{\mathrm{i}(\mu\alpha+\nu\beta)}\, \mathrm{e}^{-\frac{1}{2}\rho} \rho^{(\nu-\mu+1)/2} L_\mu^{(\nu-\mu)}(\rho) \qquad \rho = \gamma r \tag{4.12}$$

where

$$N_{\mu,\nu} = \left(\frac{\gamma\mu!}{(\mu+\nu+1)\nu!}\right)^{1/2}. \tag{4.13}$$

When acting on such functions, the w(1) generators $A_\pm$ of equation (3.20) become

$$A_\pm = \mathrm{e}^{\pm\mathrm{i}\alpha}\sqrt{\rho}\left[\pm\frac{\partial}{\partial\rho} + \frac{1}{2\rho}\left(\mathrm{i}\frac{\partial}{\partial\alpha} - \mathrm{i}\frac{\partial}{\partial\beta} \mp 1\right) - \frac{1}{2}\right] \tag{4.14}$$

and similarly for the generators $B_\pm$ of equation (3.21).

It is now straightforward to show that

$$A_\pm\Psi_{\mu,\nu} = \left(\frac{(\mu+\nu+1\pm1)\left(\mu+\frac{1}{2}\pm\frac{1}{2}\right)}{\mu+\nu+1}\right)^{1/2}\Psi_{\mu\pm1,\nu} \tag{4.15}$$

$$B_\pm\Psi_{\mu,\nu} = -\left(\frac{(\mu+\nu+1\pm1)\left(\nu+\frac{1}{2}\pm\frac{1}{2}\right)}{\mu+\nu+1}\right)^{1/2}\Psi_{\mu,\nu\pm1}. \tag{4.16}$$

The operators $A_\pm$ (respectively $B_\pm$) change n, L and Z to $n \pm \frac{1}{2}$, $L \mp \frac{1}{2}$ (respectively $L \pm \frac{1}{2}$) and $Z' = Z\left(n \pm \frac{1}{2}\right)/n$, while leaving the energy unchanged. It should be stressed that the resulting extended eigenfunctions do not correspond to eigenfunctions of any physical Coulomb problem since for the latter, n and L are restricted to integer values.

The usefulness of $A_\pm$ and $B_\pm$ appears when considering their bilinear products, whose action gives rise to physical eigenfunctions, since the integer character of all quantum numbers is then retrieved (provided Z/n is an integer). Such bilinear products generate an sp(4, $\mathbb{R}$) Lie algebra, which has various physically-relevant *su*(1, 1) or u(2) subalgebras. This type of algebraic description of the Coulomb problem could alternatively be derived from a similar description of the radial oscillator problem [10] and the known mapping of the eigenstates of the Coulomb problem onto the even angular momentum eigenstates of the four-dimensional oscillator [11].

5. Conclusion

In the present paper, we have completed the analysis of the possibilities of connections among different factorization types that we had started in I. It is worth stressing that although some of them have been suggested by Infeld and Hull [2] and analysed to a certain extent by other authors [12], for the first time we have explored them in a general and systematic way, while providing an algebraization of ladder operators.

More specifically, we have shown that the approach followed in I and in the present comment leads to a unified Lie algebraic description of type E and F factorizable Hamiltonians. The main conclusion of such an analysis is that the existence of satellite algebras is a characteristic property of type E factorizable problems. A similar construction procedure applied to type F factorization problems indeed leads to operators that do not change the energy eigenvalue and which therefore generate other types of physically relevant algebras.

This has been illustrated with a detailed discussion of the Coulomb problem, where we obtained the explicit action of the algebra generators resulting from the connection of type F factorization with either type B or type C factorization.

As a final point, it is worth mentioning that type D factorization has not been used in our construction of algebras. Its precise relationship with other factorization types indeed remains an unsettled issue (see also [12]).

References

[1] Del Sol Mesa A and Quesne C 2000 *J. Phys. A: Math. Gen.* **33** 4059
[2] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
[3] Cariñena J F and Ramos A 1999 Riccati equation, factorization method and shape invariance *Preprint* math-ph/9910020
[4] Del Sol Mesa A, Quesne C and Smirnov Yu F 1998 *J. Phys. A: Math. Gen.* **31** 321
[5] Alhassid Y, Gürsey F and Iachello F 1983 *Ann. Phys., NY* **148** 346
Alhassid Y, Gürsey F and Iachello F 1986 *Ann. Phys., NY* **167** 181
[6] Frank A and Wolf K B 1984 *Phys. Rev. Lett.* **52** 1737
Frank A and Wolf K B 1985 *J. Math. Phys.* **26** 973
[7] Schiff L I 1955 *Quantum Mechanics* (New York: McGraw-Hill)
[8] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
[9] Haymaker R W and Rau A R P 1986 *Am. J. Phys.* **54** 928
[10] Fernández D J, Negro J and del Olmo M A 1996 *Ann. Phys., NY* **252** 386
[11] Nouri S 1999 *Phys. Rev.* A **60** 1702
[12] Gangopadhyaya A, Panigrahi P K and Sukhatme U P 1994 *Helv. Phys. Acta* **67** 363
Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267